Explicit Computations of the Frozen Boundaries of Rhombus Tilings

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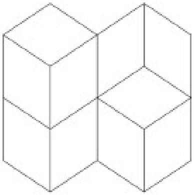
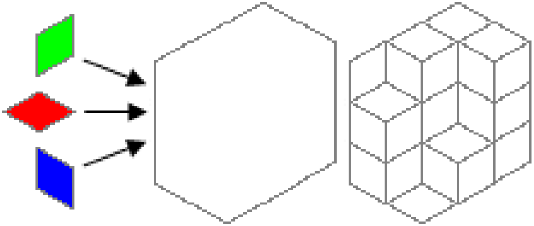
Tiling Models

Definition

A *tile* is a 60° rhombus, also known as a *lozenge*. A *tiling* is a covering of a polygonal domain with tiles such that there are no holes nor overlaps.



Tiling Example 1



Tiling Example 2

Tiling Models

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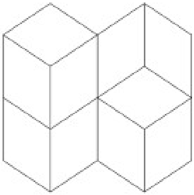

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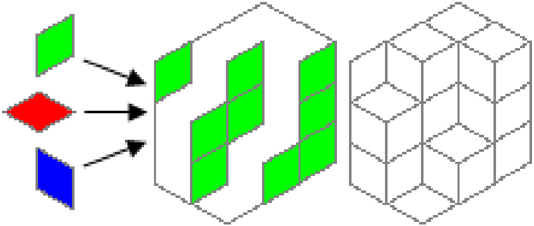

Tiling Example 2

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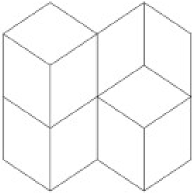
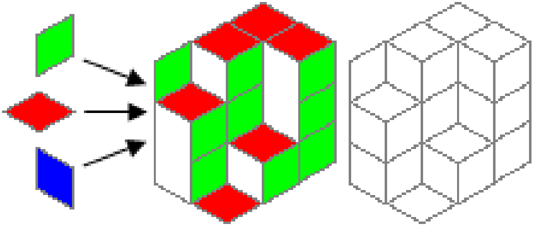


Tiling Example 1



Tiling Example 2

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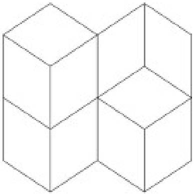

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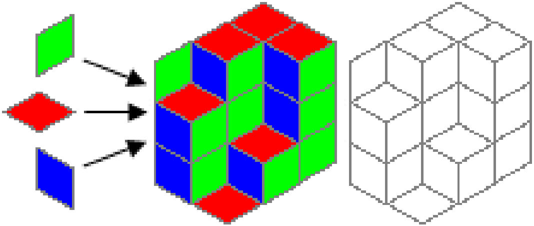
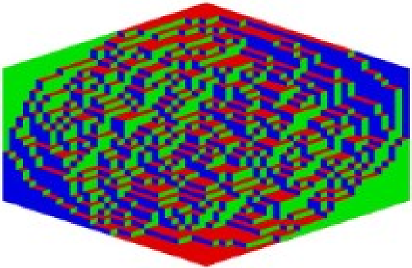
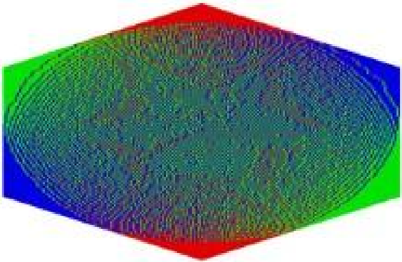


Tiling Example 2

Tiling Models



Tiling of a Hexagon

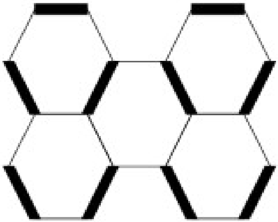


Tiling of a Hexagon with Smaller Tiles

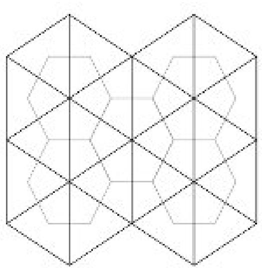
Tiling Models and Perfect Matchings

Definition

A *perfect matching* of a hexagonal lattice G is defined as a subset of edges in G that covers each vertex exactly once.

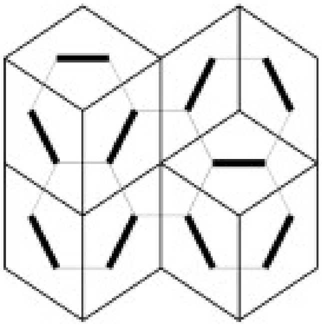


Perfect Matching



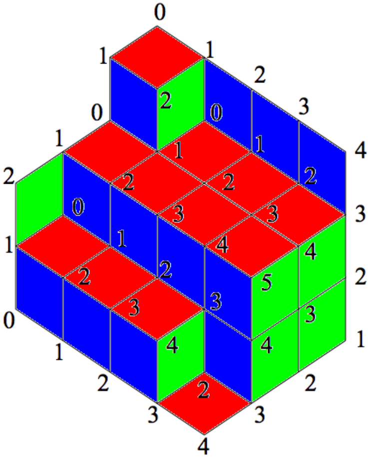
Dual Graph

Tiling Models and Perfect Matchings



Bijection Between Tiling Models
and Perfect Matchings

Tiling Models and the Height Function

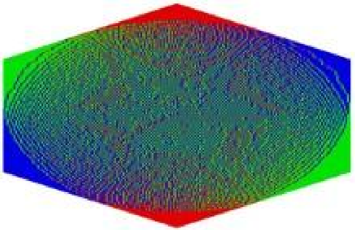


Height Model

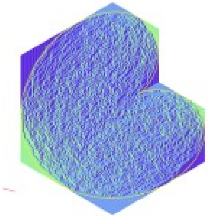
The Frozen Boundary

Theorem

Let Ω be tilable, connected polygon with 3d sides. Fix $\epsilon \geq 0$. Consider the tilings of Ω by rhombi of size $\frac{1}{N}$. Then for sufficiently large N all but an ϵ fraction of the domino tilings will have a temperate zone whose boundary stays uniformly within distance ϵ of the inscribed curve.



Frozen Boundary of a Tiling of a Hexagonal Domain



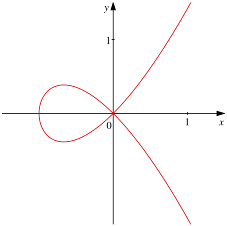
Frozen Boundary of a Tiling of an Octagonal Domain

Rational Parametrization

A *rational parametrization* of a curve is a parametrization such that $x(t)$ and $y(t)$ are both represented in the form $\frac{P(t)}{Q(t)}$, where $P(t)$ and $Q(t)$ are polynomials in t.
Example:

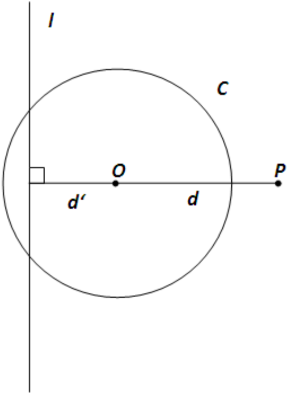
$$x(t) = -\frac{1-t^2}{1+t^2}$$

$$y(t) = -\frac{t-t^3}{1+t^2}$$



Nodal Cubic

Duality



Reciprocation over the Unit Circle

Duality

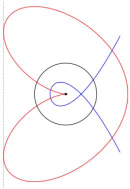
Definition

Let C be an algebraic curve. Then the *dual curve* C^* is defined as the set of poles of all the tangent lines to C.

- If C is given by parametric equations $(u(t), v(t))$, C^* has parametric equations

$$\left(\frac{v'(t)}{u'(t)v(t) - v'(t)u(t)}, \frac{-u'(t)}{u'(t)v(t) - v'(t)u(t)}\right).$$

- If C is given by the homogeneous function $f(x, y, z) = 0$, then the dual curve C^* is given by the set of lines $\left(\frac{\partial f}{\partial x}(a, b, c) : \frac{\partial f}{\partial y}(a, b, c) : \frac{\partial f}{\partial z}(a, b, c)\right)$ for every line $(a : b : c)$ in C.



A Curve and its Dual

Duality

Theorem

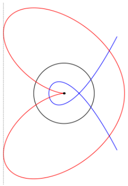
The dual of a dual curve is the original curve. That is, for any algebraic curve C, $(C^*)^* = C$.

Theorem

(Plucker's Formula) *If C has degree d, then the degree d' of C^* is given by*

$$d' = d(d-1) - 2\delta - 3\kappa,$$

where δ is the number of ordinary double points of C and κ is the number of cusps of C.



Cusp and Ordinary Double Point on Curves

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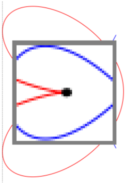

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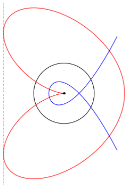

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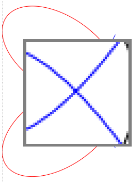

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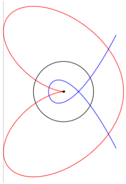


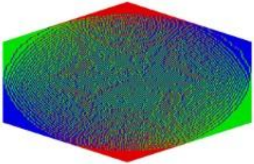
Cusp and Ordinary Double Point on Curves

Theorem Concerning the Dual of the Curve that is the Frozen Boundary

Theorem

For a 3d-gonal, tilable, polygonal domain, the frozen boundary is a rational algebraic curve whose dual has degree d.

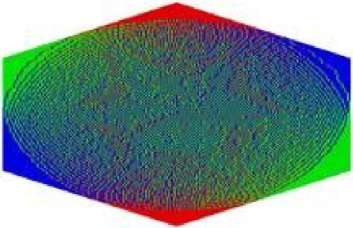
- For an n-gonal domain, if n is not divisible by 3, we choose the lowest number $3d$ greater than n. The degree of the dual curve in this case is then d.



Frozen Boundary of a Tiling of a Hexagonal Domain

Frozen Boundary of a Rhombus Tiling of a Hexagon

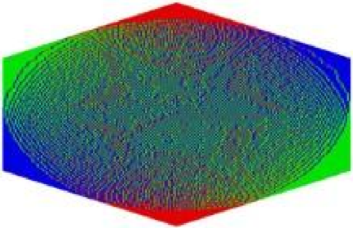
- The hexagon we are considering has 3 pairs of equal parallel sides.



Frozen Boundary of a Tiling of a Hexagonal Domain

Frozen Boundary of a Rhombus Tiling of a Hexagon

- The hexagon we are considering has 3 pairs of equal parallel sides.
- Both the inscribed curve and the dual to the inscribed curve are conics.



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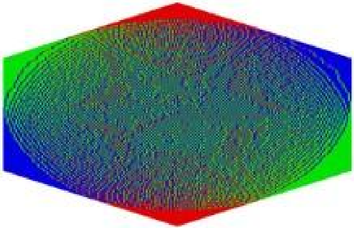
Frozen Boundary of a Rhombus Tiling of a Hexagon

- The hexagon we are considering has 3 pairs of equal parallel sides.
- Both the inscribed curve and the dual to the inscribed curve are conics.
- The inscribed curve is specifically an ellipse.

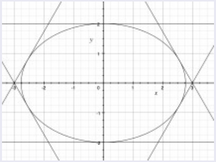


Frozen Boundary of a Tiling of a Hexagonal Domain

Frozen Boundary of a Rhombus Tiling of a Hexagon

Example 1

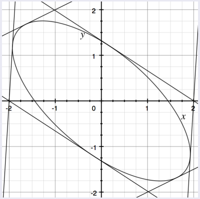
- Equations of Sides
  - $y = -\sqrt{3}(x-3)$
  - $y = \sqrt{3}(x-3)$
  - $y = -\sqrt{3}(x+3)$
  - $y = \sqrt{3}(x+3)$
  - $y = -2$
  - $y = 2$
- Equation of Frozen Boundary
  - $x^2 + 1.9166667y^2 - 7.66667 = 0$



Frozen Boundary of a Tiling of a Hexagonal Domain

Example 2

- Equations of Sides
  - $y = 0.5x + 2.5$
  - $y = 0.5x - 2.5$
  - $y = 16.66(x + 2)$
  - $y = 16.66(x - 2)$
  - $y = -.66(x + 2)$
  - $y = -.66(x - 2)$
- Equation of Frozen Boundary
  - $-0.22254026037x^2 - 0.268822y^2 + 0.465063686975 - 0.32382566942xy = 0$

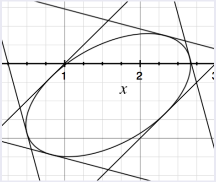


Frozen Boundary of a Tiling of a Hexagonal Domain

Frozen Boundary of a Rhombus Tiling of a Hexagon

Example 3

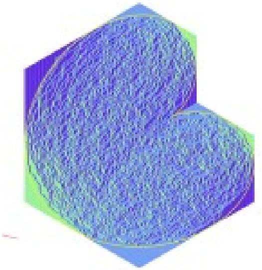
- Equations of Sides
  - $y = x - 1$
  - $y = x - 3$
  - $y = -.26795x - 1$
  - $y = -.26795x + 1$
  - $y = -3.73205(x - .25)$
  - $y = -3.73205(x - 2.6782)$
- Equation of Frozen Boundary
  - $-0.404941711057x^2 - 0.7087299025y^2 - 1.110655775625 + 0.518828854699998xy + 1.496749379050002x - 1.4174623775y = 0$



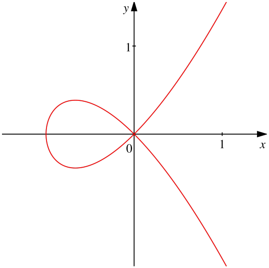
Frozen Boundary of a Tiling of a Hexagonal Domain

Frozen Boundary of a Rhombus Tiling of an Octagon

- The octagon we are considering is shown below, with seven 120° angles and one 240° angle.



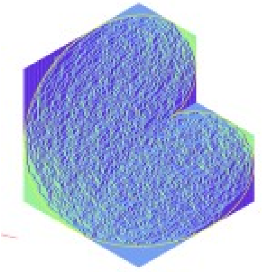
Frozen Boundary of a Tiling of an Octagonal Domain



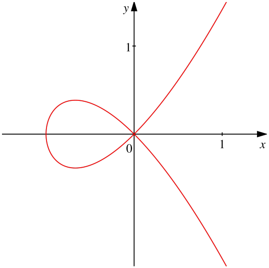
Nodal Cubic

Frozen Boundary of a Rhombus Tiling of an Octagon

- The octagon we are considering is shown below, with seven 120° angles and one 240° angle.
- The inscribed curve is a cardioid, and the dual to the inscribed curve is a nodal cubic.



Frozen Boundary of a Tiling of an Octagonal Domain



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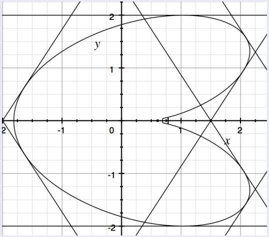
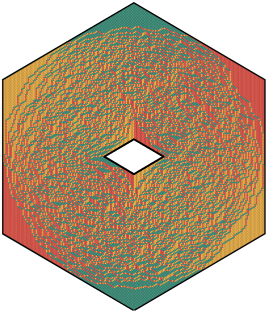
- Equations of Sides
  - $y = -\sqrt{3}(x+2)$
  - $y = \sqrt{3}(x+2)$
  - $y = -\sqrt{3}(x-3)$
  - $y = \sqrt{3}(x-3)$
  - $y = -\sqrt{3}(x-1.5)$
  - $y = \sqrt{3}(x-1.5)$
  - $y = -2$
  - $y = 2$



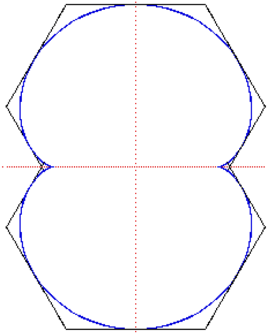
Frozen Boundary of a Tiling of a Hexagonal Domain

- Equation of Frozen Boundary
  - $-819.68 + 5255.08x - 10097.5x^2 + 2939.42x^3 + 5654.49x^4 - 126470y^2 - 47651.6xy^2 + 20749x^2y^2 + 38224.3y^4 = 0$

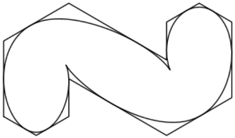
Future Directions



Hexagon with a Hole



Nephroid



More Complex Domain

Acknowledgements

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- Professor Gorin
- Dr. Khovanova
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- Dr. Gerovitch
- Dr. Etingof
- My Parents

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Andrei Okounkov

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Vadim Gorin

Random Lozenge Tilings

Vadim Gorin's professional homepage

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